MTH 301: Group Theory

Homework II

(Due 05/09)

1. Let G be a finite group.
   (a) Show that if $g^m = 1$, for some $g \in G$, then $o(g) \mid m$.
   (b) Show that if $|G|$ is a prime number, then G has to be cyclic.
2. Let G be a group.
   (a) Show that if G has no nontrivial subgroups, then G has to be of finite prime order, and hence cyclic.
   (b) Show that if $(ab)^2 = a^2b^2$ for every $a, b \in G$, then G is a abelian.
3. Let G be a group and $H \leq G$. Then show that the following sets form subgroups of G.
   (a) The set $gHg^{-1} = \{ghg^{-1} \mid h \in H\}$.
   (b) The set $Z(G) = \{g \in G \mid gx = xg, \forall x \in G\}$ called the *center of* G.
   (c) The set $C(H) = \{g \in G \mid gh = hg, \forall h \in H\}$ called the *centralizer of* H *in* G.
   (d) The set $N(H) = \{g \in G \mid gHg^{-1} = H\}$ called the *normalizer of* H *in* G.
4. Let G be a cyclic group.
   (a) Show that if $H \leq G$, then H has to be cyclic.
   (b) If $|G| = n$, then how many generators can G have.
5. Let $U_n = \{[g] \in \mathbb{Z}_n \mid \gcd(g, n) = 1\}$.
   (a) If $\cdot$ denotes multiplication modulo n, show that $(U_n, \cdot)$ is a group.
   (b) Show that U_9 is cyclic, while U_{20} is not a cyclic group.
6. Let G be a group and $H \leq G$. Then the number of distinct left (or right) cosets of H in G is called the *index of* H *in* G, denoted by $|G : H|$.
   (a) Show that if G is finite, then $|G : H| = |G|/|H|$.
   (b) Let $G = \mathbb{Z}$ and $H = m\mathbb{Z}$, for some $m \in \mathbb{Z}$. Then compute G/H and $|G : H|$.